



Titchmarsh theorems for Fourier transforms of Hölder–Lipschitz functions on compact homogeneous manifolds

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Abstract

In this paper we extend classical Titchmarsh theorems on the Fourier transform of Hölder–Lipschitz functions to the setting of compact homogeneous manifolds. As an application, we derive a Fourier multiplier theorem for L^2 -Hölder–Lipschitz spaces on compact Lie groups. We also derive conditions and a characterisation for Dini–Lipschitz classes on compact homogeneous manifolds in terms of the behaviour of their Fourier coefficients.

Keywords Compact homogeneous manifolds · Compact Lie groups · Lipschitz functions

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1 Introduction

The studies of the convergence and of the rate of decay of Fourier coefficients are among the most classical problems in Fourier analysis. Starting from the Riemann–Lebesgue theorem relating the integrability of a function on the torus \mathbb{T}^1 and the convergence of its Fourier coefficients, through the Hausdorff–Young inequality relating the integrability of a function and of its Fourier transform. Furthermore one can relate the smoothness of a function on the torus and the rate of decay of its Fourier coefficients: indeed if $f \in C^k(\mathbb{T}^1)$ for some $k \geq 1$, one has

$$\widehat{f}(j) = o(|j|^{-k}),$$

that is, $|j|^k |\widehat{f}(j)| \rightarrow 0$ as $j \rightarrow \infty$. An extension of such statement below the index $k = 1$ can be obtained in the setting of Hölder regularity. If f satisfies the Hölder condition of order $0 < \alpha \leq 1$ then

$$\widehat{f}(j) = O(|j|^{-\alpha}).$$

This time we only get big-O (Landau's notation). For the case $\alpha > \frac{1}{2}$ this estimate can be improved as a consequence of the classical Bernstein theorem (see Bernstein [3] or Zygmund [44, vol. I, p. 240]): if f satisfies the Hölder condition of order $\alpha > \frac{1}{2}$ then its Fourier series converges absolutely. It was also established by Bernstein that the index $\frac{1}{2}$ is sharp. An introduction to the classical topics discussed above can be found in [34].

The Hausdorff–Young inequality on a locally compact abelian group G_1 states that $\widehat{f} \in L^q$ if $f \in L^p(G_1)$ for $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. In [37], Edward Charles Titchmarsh studied the problem of how much this fact can be strengthened if f additionally satisfies a Lipschitz condition in the case of $G_1 = \mathbb{R}$. In fact, Titchmarsh showed that the information on \widehat{f} can be significantly improved. In [39,41,43], M. Yonis extended the Titchmarsh theorems to the case of the circle and to the very specific case of compact groups, namely, the 0-dimensional groups. Other studies of Lipschitz conditions have been done in [19] in term of the modulus of continuity. However, our aim in this paper is to derive *suitable characterisations in terms of the Fourier coefficients of the functions*.

The aim of this paper is to extend the classical Titchmarsh theorems to the setting of general compact homogeneous manifolds. In particular, this includes higher dimensional tori, compact Lie groups, real, complex or quaternionic spheres, projective spaces, and many others. As an application of such an extension, we derive a Fourier multiplier theorem for L^2 -Lipschitz spaces.

The analysis of various spaces and functional inequalities on compact Lie groups and compact homogeneous manifolds, despite its own interest and long history, has also a range of applications linking the local properties of functions with the global spectral properties of the group and to its representation theory, see e.g. Pesenson [21] for Bernstein–Nikolskii inequalities and links to the approximation theory, [22] for links to frames construction and Geller and Pesenson [17] for Besov spaces, Kushpel [18] for cubature formulae, [5] for Gevrey spaces with further links to weakly hyper-

bolic partial differential equations involving Laplacian and sub-Laplacian, [20] for Nikolskii inequality and Besov spaces, as well as further references in these papers. The convergence of the Fourier series of functions on compact Lie groups has a long history, see e.g., to mention only very few, Taylor [36], Sugiura [35], and some review in Faraut's book [12]. We can also refer to [2] for conditions for Lipschitz functions on compact totally disconnected groups, [23] on symmetric spaces of rank 1, [25] for Fourier series on Vilenkin groups (compact, metrisable, 0-dimensional, abelian groups), and results for Lipschitz multipliers in [4, 24, 26, 27] on compact 0-dimensional, locally compact abelian, and abelian 0-dimensional groups, respectively.

We note that for convolution operators (on compact Lie groups) the rate of decay of Fourier coefficients of the kernel is also related to the Schatten–von Neumann properties of the operator, see e.g. [6, 8].

We also note that anisotropic Hölder spaces (with respect to the group structure) have been also analysed on the Heisenberg group in [16] (see also [15]) with applications to partial differential equations, and some Fourier transform description was considered in [42]. Anisotropic Hölder spaces on more general Carnot groups were recently applied in [29] to questions in the potential theory.

Thus, the first Titchmarsh theorem that we deal with (Titchmarsh [37, Theorem 84]) in its version on the torus ([39, Theorem 2.6]) is recalled as Theorem A below. First, for $0 < \alpha \leq 1$ and $1 < p \leq \infty$, we define the space $\text{Lip}_{\mathbb{T}^1}(\alpha; p)$ by

$$\text{Lip}_{\mathbb{T}^1}(\alpha; p) := \left\{ f \in L^p(\mathbb{T}^1) : \|f(h + \cdot) - f(\cdot)\|_{L^p(\mathbb{T}^1)} = O(h^\alpha) \text{ as } h \rightarrow 0 \right\}. \quad (1.1)$$

Theorem A *Let $0 < \alpha \leq 1$ and $1 < p \leq 2$. If $f \in \text{Lip}_{\mathbb{T}^1}(\alpha; p)$, then $\widehat{f} \in \ell^\beta(\mathbb{Z})$ for $\frac{p}{p+\alpha p-1} < \beta \leq q$, $\frac{1}{p} + \frac{1}{q} = 1$.*

In order to illustrate the improvement in the conclusion $\widehat{f} \in \ell^\beta(\mathbb{Z})$ in Theorem A in comparison to the Hausdorff–Young theorem, consider for instance $f \in \text{Lip}_{\mathbb{T}^1}(\frac{1}{2}; 2)$, then one has $\frac{p}{p+\alpha p-1} = 1$, and thus $\widehat{f} \in \ell^\beta(\mathbb{Z})$ for all $\beta \in (1, 2]$.

In Theorem A' and Theorem 3.2 we extend Theorem A to the setting of general compact homogeneous manifolds, also slightly sharpening the conclusion.

We point out that our proof of Theorem A' (through Theorem 3.2) for compact homogeneous manifolds also simplifies the one by Younis in some aspects already on the circle \mathbb{T}^1 , particularly avoiding the use of combinatorial arguments for summations by parts. The fact that the lower bound $\frac{p}{p+\alpha p-1}$ is sharp can be proved by means of the function

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{in \log n}}{n^{\frac{1}{2}+\alpha}} e^{inx} \quad (0 < \alpha < 1)$$

introduced by Hardy and Littlewood. Indeed, one has $f \in \text{Lip}_{\mathbb{T}^1}(\alpha; 2)$ but $\widehat{f} \notin \ell^{\frac{2}{2\alpha+1}}(\mathbb{Z})$ (see Zygmund [44, vol 1, p. 243]).

The second Titchmarsh theorem (Titchmarsh [37, Theorem 85]) gives the necessary and sufficient conditions on the remainder of the series of Fourier coefficients for L^2 -

Lipschitz functions, i.e. for the case $p = 2$. In the case of the circle ([39, Theorem 2.17]) it can be stated as follows:

Theorem B Let $0 < \alpha \leq 1$ and $f \in L^2(\mathbb{T}^1)$. Then $f \in \text{Lip}_{\mathbb{T}^1}(\alpha; 2)$ if and only if

$$\sum_{|j| \geq N} |\widehat{f}(j)|^2 = O(N^{-2\alpha}) \text{ as } N \rightarrow \infty.$$

We will now briefly outline our main results in the setting of general compact homogeneous manifolds. If K is a closed subgroup of a compact Lie group G , the quotient space G/K can be canonically identified with an analytic manifold M . Henceforth M will denote a compact homogeneous manifold. In the case when $K = \{e\}$ is the identity of the group G , we have $M = G$ is a compact Lie group itself. To simplify the notation in the sequel we will identify representations of G with their equivalence classes.

We denote by \widehat{G} the unitary dual of G , and by \widehat{G}_0 the subset of \widehat{G} consisting of representations ξ of class I with respect to the subgroup K : this means that $[\xi] \in \widehat{G}_0$ if ξ has at least one non-zero invariant vector with respect to K , i.e. that for some non-zero a in the representation space of ξ we have $\xi(h)a = a$ for all $h \in K$. We will also denote by k_ξ the dimension of the space of invariant vectors of the representation ξ with respect to K .

In particular, if $K = \{e\}$ so that $G/K = G$ is a compact Lie group, then $k_\xi = d_\xi$ is the dimension of the representation $\xi \in \widehat{G}$. If the subgroup K is massive, that is if the dimension of the space of invariant vectors of the representation ξ is one for each class I representation, then we have $k_\xi = 1$ for all $\xi \in \widehat{G}_0$, and $k_\xi = 0$ for all $\xi \in \widehat{G} \setminus \widehat{G}_0$.

In order to extend the Definition (1.1) on the circle to a compact homogeneous manifold $M = G/K$ we will employ the geodesic distance on G . For the distance from the ‘unit’ element e to $h \in G$ we will write $|h| = d(h, e)$. This convention is not restrictive since the analysis of our problems is local.

Definition 1.1 Let $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$. We define the space $\text{Lip}_{G/K}(\alpha; p)$ by

$$\text{Lip}_{G/K}(\alpha; p) := \{f \in L^p(G/K) : \|f(h \cdot) - f(\cdot)\|_{L^p(G/K)} = O(|h|^\alpha) \text{ as } |h| \rightarrow 0\},$$

that is, by the condition that

$$\int_{G/K} |f(hx) - f(x)|^p dx = O(|h|^{\alpha p}) \quad \text{as } |h| \rightarrow 0, \quad (1.2)$$

for $1 \leq p < \infty$, with a natural modification for $p = \infty$.

The space $\text{Lip}_{G/K}(\alpha; p)$ endowed with the norm

$$\|f\|_{\text{Lip}_{G/K}(\alpha; p)} := \|f\|_{L^p(G/K)} + \sup_{|h| \neq 0} |h|^{-\alpha} \|f(h \cdot) - f(\cdot)\|_{L^p(G/K)}$$

becomes a Banach space.

If $f \in \text{Lip}_{G/K}(\alpha; p)$ we say that f is $(\alpha; p)$ -Lipschitz. Since G acts on G/K from left, it is natural to consider left shifts $f(h \cdot)$ with respect to h in Definition 1.1. However, if $K = \{e\}$ and our homogeneous space is the Lie group G itself, one can also consider right shifts by h , and to define the right-Lipschitz space by taking $f(\cdot h)$ instead of $f(h \cdot)$ in the Definition 1.1, namely, by the condition

$$\int_{G/K} |f(xh) - f(x)|^p dx = O(|h|^{\alpha p}) \text{ as } |h| \rightarrow 0, \quad (1.3)$$

for $1 \leq p < \infty$, with a natural modification for $p = \infty$.

Some remarks regarding the equivalence between these two notions in the case of groups are included at the end of Sect. 2 in Remark 2.3.

On the other hand, for $1 \leq p \leq \infty$ we can also define p -Lipschitz spaces on G via localisations. Given $f \in L^p(G/K)$, we say that

$$f \in \widetilde{\text{Lip}}_{G/K}(\alpha; p) \text{ if } \chi^*(\chi f) \in \text{Lip}_{\mathbb{R}^n}(\alpha; p) \quad (1.4)$$

for all coordinate chart cut-off functions χ , where χ^* denotes the pull-back by χ . We can observe that for $p = \infty$, with natural modifications of the definitions, we have

$$\widetilde{\text{Lip}}_{G/K}(\alpha; \infty) = \text{Lip}_{G/K}(\alpha; \infty). \quad (1.5)$$

In view of Lemma 2.4, it is enough to show this for compact Lie groups. The inclusion $\widetilde{\text{Lip}}_{G/K}(\alpha; \infty) \subset \text{Lip}_{G/K}(\alpha; \infty)$ is obvious due to the compactness of G/K . The converse also follows if we take $h = \exp(h_j X_j)$, $j = 1, \dots, \dim G$, for small real numbers h_j , with $\exp : \mathfrak{g} \rightarrow G$ the exponential mapping near e , and $\{X_j\}_{j=1}^{\dim G}$ a basis of \mathfrak{g} , e.g. orthonormalised with respect to the Killing form. Using that the push-forwards χ^* by coordinate chart maps have non-vanishing Jacobians, compactness, and adapting the argument above in the other direction, the equality (1.5) can be extended to any p , namely, we have

$$\widetilde{\text{Lip}}_{G/K}(\alpha; p) = \text{Lip}_{G/K}(\alpha; p), \quad 1 \leq p \leq \infty. \quad (1.6)$$

The main results of this work are Theorem A' and Theorem B' below, extending the statements of Theorem A and Theorem B, respectively, from the circle to compact homogeneous manifolds G/K .

Theorem A'. *Let G/K be a compact homogeneous manifold of dimension n . Let $0 < \alpha \leq 1$, $1 < p \leq 2$, and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in \text{Lip}_{G/K}(\alpha; p)$. Then*

$$\widehat{(I - \mathcal{L}_{G/K})^{\frac{1}{2}} f} \in \ell^\beta(\widehat{G}_0) \text{ for } \frac{n}{\alpha + n - \frac{n}{p} - 1} \leq \beta \leq q,$$

where $\mathcal{L}_{G/K}$ is the Laplacian on G/K . Consequently, we also have

$$\widehat{f} \in \ell^\gamma(\widehat{G}_0) \text{ for } \frac{np}{\alpha p + np - n} < \gamma \leq q.$$

Here for a sequence $\sigma(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ the norm $\ell^\beta(\widehat{G}_0)$ for $1 \leq \beta < \infty$ is defined by

$$\|\sigma\|_{\ell^\beta(\widehat{G}_0)} := \sum_{[\xi] \in \widehat{G}_0} d_\xi k_\xi \left(\frac{\|\sigma(\xi)\|_{\text{HS}}}{\sqrt{k_\xi}} \right)^\beta, \quad (1.7)$$

where $\|\sigma(\xi)\|_{\text{HS}} = \sqrt{\text{Tr}(\sigma(\xi)\sigma(\xi)^*)}$ is the usual Hilbert–Schmidt norm. This is a natural family of spaces on \widehat{G}_0 , extending the ℓ^2 -norm appearing in the Plancherel identity in (1.9), also satisfying the Hausdorff–Young inequalities on G/K and many other functional analytic properties. These spaces were analysed in [20, Section 2] in the setting of compact homogeneous manifolds, extending the corresponding definition on the compact Lie groups introduced earlier in [30, Section 10.3.3], in which case we would have $\widehat{G}_0 = \widehat{G}$ and $k_\xi = d_\xi$: see Definition 2.2 for this case. The Fourier coefficients of functions on G/K , although matrices in $\mathbb{C}^{d_\xi \times d_\xi}$, have a lot of zeros, see (2.13) and (2.14). This explains the factor $\sqrt{k_\xi}$ appearing in the denominator in (1.7). The spaces $\ell^\beta(\widehat{G}_0)$ are interpolation spaces, with embeddings into another known family of ℓ^β -spaces defined using Schatten norms, we refer to [20, Section 2] for a description of functional analytic properties of these spaces and to [14, Section 2.1.4] for the embedding properties.

Clearly (see Remark 3.3) Theorem A' implies Theorem A when $G = \mathbb{T}^1$ and $K = \{0\}$, so that $G/K = \mathbb{T}^1$. Since both Theorem A' and Theorem B' absorb Titchmarsh theorems in the case of the torus they are also sharp.

Theorem B'. *Let $0 < \alpha \leq 1$ and $f \in L^2(G/K)$. Then $f \in \text{Lip}_{G/K}(\alpha; 2)$ if and only if*

$$\sum_{\xi \in \widehat{G}_0, \langle \xi \rangle \geq N} d_\xi \|\widehat{f}(\xi)\|_{\text{HS}}^2 = O(N^{-2\alpha}) \text{ as } N \rightarrow \infty. \quad (1.8)$$

Here $\langle \xi \rangle$ are the eigenvalues of the elliptic operator $(I - \mathcal{L}_{G/K})^{1/2}$ corresponding to the representations $\xi \in \widehat{G}_0$, where $\mathcal{L}_{G/K}$ is the Laplacian on G/K : namely, we have $(I - \mathcal{L}_{G/K})^{1/2} \xi_{ij} = \langle \xi \rangle \xi_{ij}$ for all $\xi \in \widehat{G}_0$, $1 \leq i \leq d_\xi$ and $1 \leq j \leq k_\xi$. The choice of the range of the indices i, j happens in this way since these are the only representation coefficients interacting with the Fourier coefficients of functions on G/K , see (2.13) and (2.14). In the case of Lie groups the quantity $\langle \xi \rangle$ is also explained in detail in (2.4).

Theorem B' implies Theorem B since the Plancherel formula on G/K takes the form

$$\sum_{\xi \in \widehat{G}_0} d_\xi \|\widehat{f}(\xi)\|_{\text{HS}}^2 = \|f\|_{L^2(G/K)}^2, \quad (1.9)$$

so that the left hand side of (1.8) is the remainder of the Fourier series decomposition of $f \in L^2(G/K)$, see also (2.6).

In view of Lemma 2.4, Theorem A' and Theorem B' follow from their respective versions on compact Lie groups formulated in Theorems 3.2 and 3.4, respectively.

As an application of the obtained characterisations, we will show a Fourier multiplier theorem for Hölder/Lipschitz spaces on compact Lie groups. Here, the Fourier multiplier A with symbol $a(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ for each ξ acts by multiplication by $a(\xi)$ on the Fourier transform side, i.e. by

$$\widehat{Af}(\xi) = a(\xi)\widehat{f}(\xi).$$

The matrix $a(\xi)$ arises as a linear transformation $a(\xi) : \mathbb{C}^{d_\xi} \rightarrow \mathbb{C}^{d_\xi}$, where \mathbb{C}^{d_ξ} is the representation space of ξ after fixing some (any) basis. We denote by $\|a(\xi)\|_{op} = \|a(\xi)\|_{\mathcal{L}(\mathbb{C}^{d_\xi}, \mathbb{C}^{d_\xi})}$ the operator norm of this linear transformation, with \mathbb{C}^{d_ξ} equipped with the usual Euclidean distance. If \mathcal{L}_G is the Laplacian on G (or the Casimir element of the universal enveloping algebra), we denote by $\langle \xi \rangle$ the eigenvalue of the elliptic pseudo-differential operator $(I - \mathcal{L}_G)^{1/2}$ corresponding to ξ , i.e.

$$(I - \mathcal{L}_G)^{1/2} \xi_{ij}(x) = \langle \xi \rangle \xi_{ij}(x) \quad \text{for all } 1 \leq i, j \leq d_\xi,$$

see (2.5) for more explanation.

Corollary 1.2 *Let $0 \leq \gamma < 1$ and let $a : \widehat{G} \rightarrow \bigcup_{d \in \mathbb{N}} \mathbb{C}^{d \times d}$ be such that $a(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ for each ξ and*

$$\|a(\xi)\|_{op} \leq C \langle \xi \rangle^{-\gamma}.$$

Then the Fourier multiplier A by symbol a ,

$$A : \text{Lip}_G(\alpha; 2) \rightarrow \text{Lip}_G(\alpha + \gamma; 2)$$

is bounded for all α such that $0 < \alpha < 1 - \gamma$.

For example, the matrix symbol of the Bessel potential $A = (I - \mathcal{L}_G)^{-\gamma/2}$ is $a(\xi) = \langle \xi \rangle^{-\gamma} I_{d_\xi}$, where $I_{d_\xi} \in \mathbb{C}^{d_\xi \times d_\xi}$ is the identity matrix, so that $\|a(\xi)\|_{op} = \langle \xi \rangle^{-\gamma}$.

A similar result to Corollary 1.2 is true on homogeneous manifolds G/K , although in this case, in view of (2.13) and (2.14), the symbol $a(\xi)$ may be assumed to be zero outside the first $k_\xi \times k_\xi$ block, i.e. we would take $a(\xi) = 0$ for $\xi \notin \widehat{G}_0$, and $a(\xi)_{ij} = 0$ for all $\xi \in \widehat{G}_0$ and for $i > k_\xi$ or $j > k_\xi$.

The multiplier theorem in Corollary 1.2 complements other known multiplier theorems on compact Lie groups, e.g. the Mihlin multiplier theorem [32,33], Hörmander multiplier theorem [13], or more general results for the boundedness of pseudo-differential operators in L^p -spaces [7].

In Sect. 4 we discuss analogues of the obtained results for the Dini–Lipschitz classes. The Dini–Lipschitz classes are a logarithmic extension on Lipschitz classes and these spaces have been analysed in different settings, e.g. in low dimensions [40],

or using Helgason transforms on rank 1 symmetric spaces in [10,11]. Our analysis in this paper is based on a logarithmic extension of the Duren lemma allowing inclusion of the log-terms in the characterisations.

2 Preliminaries

In this section we recall some basic facts on the Fourier analysis and the notion of global matrix-symbols for pseudo-differential operators on compact Lie groups and more generally on compact homogeneous manifolds. We refer to [30,31] for a comprehensive account of such topics.

Since the proof of our main theorems will be reduced to the case of compact Lie groups as it will be explained at the end of this section, it will be convenient for our purposes to start with the setting of compact Lie groups. Given a compact Lie group G , we equip it with the normalised Haar measure $\mu \equiv dx$ on the Borel σ -algebra associated to the topology of the smooth manifold G . The Lie algebra of G will be denoted by \mathfrak{g} . We also denote by \widehat{G} the set of equivalence classes of continuous irreducible unitary representations of G and by $\text{Rep}(G)$ the set of all such representations. Since G is compact, the set \widehat{G} is discrete. For $[\xi] \in \widehat{G}$, by choosing a basis in the representation space of ξ , we can view ξ as a matrix-valued function $\xi : G \rightarrow \mathbb{C}^{d_\xi \times d_\xi}$, where d_ξ is the dimension of the representation space of ξ . By the Peter–Weyl theorem the collection

$$\left\{ \sqrt{d_\xi} \xi_{ij} : 1 \leq i, j \leq d_\xi, [\xi] \in \widehat{G} \right\}$$

is an orthonormal basis of $L^2(G)$. If $f \in L^1(G)$ we define its global Fourier transform at ξ by

$$\mathcal{F}_G f(\xi) \equiv \widehat{f}(\xi) := \int_G f(x) \xi(x)^* dx. \quad (2.1)$$

Thus, if ξ is a matrix representation, we have $\widehat{f}(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$. The Fourier inversion formula is a consequence of the Peter–Weyl theorem:

$$f(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}(\xi(x) \widehat{f}(\xi)). \quad (2.2)$$

Given a sequence of matrices $a(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$, we can define

$$(\mathcal{F}_G^{-1} a)(x) := \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}(\xi(x) a(\xi)), \quad (2.3)$$

where the series can be interpreted in a distributional sense or absolutely depending on the growth of (the Hilbert–Schmidt norms of) $a(\xi)$. For a further discussion we refer the reader to [30].

For each $[\xi] \in \widehat{G}$, the matrix elements of ξ are the eigenfunctions for the Laplacian \mathcal{L}_G (or the Casimir element of the universal enveloping algebra), with the same eigenvalue which we denote by $-\lambda_{[\xi]}^2$, so that

$$-\mathcal{L}_G \xi_{ij}(x) = \lambda_{[\xi]}^2 \xi_{ij}(x) \quad \text{for all } 1 \leq i, j \leq d_\xi. \quad (2.4)$$

The weight for measuring the decay or growth of Fourier coefficients in this setting is

$$\langle \xi \rangle := \left(1 + \lambda_{[\xi]}^2\right)^{\frac{1}{2}}, \quad (2.5)$$

the eigenvalues of the elliptic first-order pseudo-differential operator $(I - \mathcal{L}_G)^{\frac{1}{2}}$. The Parseval identity takes the form

$$\|f\|_{L^2(G)} = \left(\sum_{[\xi] \in \widehat{G}} d_\xi \|\widehat{f}(\xi)\|_{\text{HS}}^2 \right)^{1/2}, \quad \text{where } \|\widehat{f}(\xi)\|_{\text{HS}}^2 = \text{Tr}(\widehat{f}(\xi) \widehat{f}(\xi)^*), \quad (2.6)$$

which gives the norm on $\ell^2(\widehat{G})$.

For a linear continuous operator A from $C^\infty(G)$ to $\mathcal{D}'(G)$ we define its *matrix-valued symbol* $\sigma(x, \xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ by

$$\sigma(x, \xi) := \xi(x)^*(A\xi)(x) \in \mathbb{C}^{d_\xi \times d_\xi}, \quad (2.7)$$

where $A\xi(x) \in \mathbb{C}^{d_\xi \times d_\xi}$ is understood as $(A\xi(x))_{ij} = (A\xi_{ij})(x)$, i.e. by applying A to each component of the matrix $\xi(x)$. Then one has ([30,31]) the global quantization

$$Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \text{Tr}(\xi(x) \sigma(x, \xi) \widehat{f}(\xi)) \equiv \sigma(X, D)f(x), \quad (2.8)$$

in the sense of distributions, and the sum is independent of the choice of a representation ξ from each equivalence class $[\xi] \in \widehat{G}$. If A is a linear continuous operator from $C^\infty(G)$ to $C^\infty(G)$, the series (2.8) is absolutely convergent and can be interpreted in the pointwise sense. The symbol σ can be interpreted as a matrix-valued function on $G \times \widehat{G}$. We refer to [30,31] for the consistent development of this quantization and the corresponding symbolic calculus. If the operator A is left-invariant then its symbol σ does not depend on x . We often simply call such operators invariant.

In the sequel we will also need the Taylor expansion of functions on compact Lie groups. For $f \in C^\infty(G)$, we have

$$f(x) = \sum_{|\alpha| \leq N-1} D^{(\alpha)} f(e) q_\alpha(x) + O(|x|^N), \quad (2.9)$$

for some invariant differential operators $D^{(\alpha)}$ of order $|\alpha|$, for an admissible family of functions q_α , with $|x|$ denoting the geodesic distance from x to e , see [30, Section 10.6].

We note that Definition 1.1 in the case of compact Lie groups can be stated in the following form by taking the geodesic distance on G from the unit element e to $h \in G$:

Definition 2.1 Let G be a compact Lie group. Let $0 < \alpha \leq 1$ and $1 \leq p \leq \infty$. We define the space $\text{Lip}_G(\alpha; p)$ by

$$\text{Lip}_G(\alpha; p) := \{f \in L^p(G) : \|f(h \cdot) - f(\cdot)\|_{L^p(G)} = O(|h|^\alpha) \text{ as } |h| \rightarrow 0\},$$

that is,

$$\int_G |f(hx) - f(x)|^p dx = O(|h|^{\alpha p}) \quad \text{as } |h| \rightarrow 0,$$

for $1 \leq p < \infty$, with a natural modification for $p = \infty$.

The space $\text{Lip}_G(\alpha; p)$ endowed with the norm

$$\|f\|_{\text{Lip}_G(\alpha; p)} := \|f\|_{L^p(G)} + \sup_{|h| \neq 0} |h|^{-\alpha} \|f(h \cdot) - f(\cdot)\|_{L^p(G)}$$

becomes a Banach space for all $1 \leq p \leq \infty$.

For the analysis of the Fourier transform we will require the following function spaces on the unitary dual \widehat{G} . We refer the reader to [30, Section 10.3.3] for the basic properties of such spaces.

Definition 2.2 For $0 < p < \infty$, we will write $\ell^p(\widehat{G})$ for the space of all $H = H(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ such that

$$\|H\|_{\ell^p(\widehat{G})} := \left(\sum_{[\xi] \in \widehat{G}_0} d_\xi^{p(\frac{2}{p} - \frac{1}{2})} \|H(\xi)\|_{\text{HS}}^p \right)^{\frac{1}{p}} < \infty.$$

If $1 \leq p < \infty$ the quantity $\|H\|_{\ell^p(\widehat{G})}$ defines a norm and $\ell^p(\widehat{G})$ endowed with it becomes a Banach space. If $0 < p < 1$ we can associate a Fréchet metric and the associated space becomes a complete metric space.

We record some asymptotic properties on a compact Lie group G of dimension n , that will be of use on several occasions: asymptotically as $\lambda \rightarrow \infty$ we have

$$\sum_{\langle \xi \rangle \leq \lambda} d_\xi^2 \langle \xi \rangle^{rn} \asymp \lambda^{(r+1)n} \quad \text{for } r > -1, \quad (2.10)$$

and

$$\sum_{\langle \xi \rangle \geq \lambda} d_\xi^2 \langle \xi \rangle^{rn} \asymp \lambda^{(r+1)n} \quad \text{for } r < -1. \quad (2.11)$$

These properties follow from the Weyl spectral asymptotic formula for the elliptic pseudo-differential operator $I - \mathcal{L}_G$, and we can refer to [1] for their proof. We also note the convergence criterion

$$\sum_{\langle \xi \rangle \leq \lambda} d_{\xi}^2 \langle \xi \rangle^{-s} < \infty \text{ if and only if } s > n, \quad (2.12)$$

see [5] for the proof.

Remark 2.3 The notions of left and right-Lipschitz in (1.2) and (1.3) are related in the case of compact Lie groups in the following way. We write $i(x) = x^{-1}$ and define $f^{\cdot}(x) = f \circ i(x) = f(x^{-1})$. We note that

$$f(xh) - f(x) = f^{\cdot}(h^{-1}x^{-1}) - f^{\cdot}(x^{-1})$$

and since $|h| \simeq |h^{-1}|$ we have obtained that f is a right-Lipschitz of type $(\alpha; p)$ function if and only if f^{\cdot} is left-Lipschitz of type $(\alpha; p)$.

Our statements will be stated for left-Lipschitz $(\alpha; p)$. A look at the arguments in the proofs of our results shows that all the statements also hold for right-Lipschitz $(\alpha; p)$ type. At the level of compact Lie groups both concepts agree when $p = 2$ as a consequence of Theorem 3.5: indeed, the criterion (3.7) is the same for both left- or right-Lipschitz functions.

We shall now explain how the proof of Theorems A' and B' can be reduced to the case of compact Lie groups. For a function $f \in C^{\infty}(G/K)$, its canonical lifting \tilde{f} is defined by $\tilde{f}(x) = f(xK)$ for all $x \in G$, so that \tilde{f} is constant on the right cosets.

Lemma 2.4 *We have $\tilde{f} \in \text{Lip}_G(\alpha; p)$ if and only if $f \in \text{Lip}_{G/K}(\alpha; p)$.*

Proof A simple calculation shows

$$\int_G |\tilde{f}(hx) - \tilde{f}(x)|^p dx = \int_{G/K} |f(hy) - f(y)|^p dy$$

with equality, if the Haar measure is normalised in such a way that the measure of K is equal to one. \square

The Fourier coefficients of canonical liftings satisfy

$$\widehat{\tilde{f}}(\xi) = 0 \quad \text{for all } \xi \notin \widehat{G}_0, \quad (2.13)$$

and

$$\widehat{\tilde{f}}(\xi)_{ij} = 0 \quad \text{for all } \xi \in \widehat{G}_0, \quad i > k_{\xi}, \quad (2.14)$$

see e.g. Vilenkin [38]. We refer to [5] or [20] for more details on the Fourier analysis on compact homogeneous manifolds.

3 Titchmarsh theorems for Fourier transforms of Lipschitz functions

In this section we prove our results. We start with a lemma on Fourier transforms that we will apply in the proof of the extension of Theorem reftha. The notion of global symbol discussed in Sect. 2 will be useful in the proof of the first main result. Henceforth G will denote a compact Lie group of dimension n .

Lemma 3.1 *Let $H : \widehat{G} \rightarrow \bigcup_{d \in \mathbb{N}} \mathbb{C}^{d \times d}$ be such that $H(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ for each ξ . Let $1 \leq \beta_0 < \infty$. Then*

$$\langle \xi \rangle H(\xi) \in \ell^{\beta_0}(\widehat{G}) \implies H \in \ell^\beta(\widehat{G}),$$

for all $\frac{n\beta_0}{\beta_0+n} < \beta < \infty$.

Proof Since $\langle \xi \rangle \geq 1$, we have $\langle \xi \rangle H(\xi) \in \ell^{\beta_0}(\widehat{G})$ implies that $H \in \ell^{\beta_0}(\widehat{G})$, and hence $H \in \ell^\beta(\widehat{G})$ for all $\beta \geq \beta_0$. So we need to prove the case $\beta < \beta_0$, which we now assume.

For $\beta < \beta_0$ we can write

$$\begin{aligned} \|H\|_{\ell^\beta(\widehat{G})}^\beta &= \sum_{[\xi] \in \widehat{G}} d_\xi^2 \left(\frac{\|H(\xi)\|_{\text{HS}}}{\sqrt{d_\xi}} \right)^\beta \\ &= \sum_{[\xi] \in \widehat{G}} d_\xi^2 \langle \xi \rangle^{-\beta} \left(\frac{\langle \xi \rangle \|H(\xi)\|_{\text{HS}}}{\sqrt{d_\xi}} \right)^\beta \\ &= \sum_{[\xi] \in \widehat{G}} d_\xi^{2\frac{\beta}{\beta_0}} \left(\frac{\langle \xi \rangle \|H(\xi)\|_{\text{HS}}}{\sqrt{d_\xi}} \right)^\beta d_\xi^{2(1-\frac{\beta}{\beta_0})} \langle \xi \rangle^{-\beta} \\ &= \sum_{[\xi] \in \widehat{G}} a_\xi b_\xi, \end{aligned}$$

where $a_\xi = d_\xi^{2\frac{\beta}{\beta_0}} \left(\frac{\langle \xi \rangle \|H(\xi)\|_{\text{HS}}}{\sqrt{d_\xi}} \right)^\beta$ and $b_\xi = d_\xi^{2(1-\frac{\beta}{\beta_0})} \langle \xi \rangle^{-\beta}$. By (2.12) and the Hölder inequality applied to the last sum for the indices $\frac{\beta_0}{\beta}, \frac{\beta_0}{\beta_0-\beta}$ we obtain

$$\begin{aligned} \sum_{[\xi] \in \widehat{G}} d_\xi^2 \left(\frac{\|H(\xi)\|_{\text{HS}}}{\sqrt{d_\xi}} \right)^\beta &\leq \left(\sum_{[\xi] \in \widehat{G}} d_\xi^2 \left(\frac{\langle \xi \rangle \|H(\xi)\|_{\text{HS}}}{\sqrt{d_\xi}} \right)^{\beta_0} \right)^{\frac{\beta}{\beta_0}} \\ &\quad \times \left(\sum_{[\xi] \in \widehat{G}} d_\xi^{2(1-\frac{\beta}{\beta_0})} \frac{\beta_0}{\beta_0-\beta} \langle \xi \rangle^{-\beta(\frac{\beta_0}{\beta_0-\beta})} \right)^{(1-\frac{\beta}{\beta_0})} \end{aligned}$$

$$\begin{aligned}
&= \|\langle \cdot \rangle H(\cdot)\|_{\ell^{\beta_0}(\widehat{G})}^{\beta} \left(\sum_{[\xi] \in \widehat{G}} d_{\xi}^2 \langle \xi \rangle^{-\beta(\frac{\beta_0}{\beta_0-\beta})} \right)^{(1-\frac{\beta}{\beta_0})} \\
&\leq C \left(\sum_{[\xi] \in \widehat{G}} d_{\xi}^2 \langle \xi \rangle^{-\beta(\frac{\beta_0}{\beta_0-\beta})} \right)^{(1-\frac{\beta}{\beta_0})} < \infty,
\end{aligned}$$

provided that $\frac{\beta\beta_0}{\beta_0-\beta} > \dim G = n$. For the convergence of the series in the last inequality we have used (2.12). To finish the proof we just note that the condition $\frac{n\beta_0}{\beta_0+n} < \beta$ is equivalent to $\frac{\beta\beta_0}{\beta_0-\beta} > n$. \square

We now state our first main result which extends Theorem A to general compact Lie groups and consequently, in view of Lemma 2.4, to compact homogeneous manifolds.

Theorem 3.2 *Let G be a compact Lie group of dimension n . Let $0 < \alpha \leq 1$, $1 < p \leq 2$, and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in \text{Lip}_G(\alpha; p)$. Then*

$$\widehat{(I - \mathcal{L}_G)^{\frac{1}{2}} f} \in \ell^{\beta}(\widehat{G}) \text{ for } \frac{n}{\alpha + n - \frac{n}{p} - 1} \leq \beta \leq q.$$

Consequently,

$$\widehat{f} \in \ell^{\gamma}(\widehat{G}) \text{ for } \frac{np}{\alpha p + np - n} < \gamma \leq q.$$

Proof For $f \in L^p(G)$ we will denote by f_h the function $f_h(x) = f(hx)$. We first note that

$$\begin{aligned}
\widehat{f_h}(\xi) &= \int_G f(hx) \xi(x)^* dx \\
&= \int_G f(hx) \xi(hx)^* \xi(h) dx \\
&= \widehat{f}(\xi) \xi(h).
\end{aligned}$$

Hence the Fourier transform of $f_h - f$ is given by

$$\mathcal{F}(f_h - f)(\xi) = \widehat{f}(\xi)(\xi(h) - I_{d_{\xi}}),$$

where $I_{d_{\xi}}$ is the identity matrix in $\mathbb{C}^{d_{\xi} \times d_{\xi}}$. By the Hausdorff–Young inequality we get

$$\|\widehat{f}(\xi)(\xi(h) - I_{d_{\xi}})\|_{\ell^q(\widehat{G})}^q \leq \|f_h - f\|_{L^p(G)}^q = O(|h|^{\alpha q}). \quad (3.1)$$

We now estimate from below $\|\widehat{f}(\xi)(\xi(h) - I_{d_{\xi}})\|_{\ell^q(\widehat{G})}^q$. To do so we start with the terms $\|\widehat{f}(\xi)(\xi(h) - I_{d_{\xi}})\|_{\text{HS}}^q$ for every ξ according to the definition of the norm $\|\cdot\|_{\ell^q(\widehat{G})}$.

We observe that

$$\begin{aligned}\|\widehat{f}(\xi)(\xi(h) - I_{d_\xi})\|_{\text{HS}}^2 &= \text{Tr} \left(\widehat{f}(\xi)(\xi(h) - I_{d_\xi})(\xi(h)^* - I_{d_\xi}^*) \widehat{f}(\xi)^* \right) \\ &= \text{Tr} \left((\xi(h) - I_{d_\xi})(\xi(h)^* - I_{d_\xi}^*) \widehat{f}(\xi)^* \widehat{f}(\xi) \right).\end{aligned}\quad (3.2)$$

We now apply Taylor expansions of type (2.9) to the terms $(\xi(h) - I_{d_\xi})$, $(\xi(h)^* - I_{d_\xi}^*)$. For each entry in the matrices they are of the form

$$\xi(h)_{ij} - \delta_{ij} = \sum_{|\alpha|=1} D^{(\alpha)} \xi_{ij}(e) q_\alpha(h) + O(|h|^2),$$

where $q_\alpha \in C^\infty(G)$ vanishes at e of order 1 and e is its isolated zero, i.e., there exist constants $C_1, C_2 > 0$ such that

$$C_1|y| \leq |q_\alpha(y)| \leq C_2|y|.$$

In the matrix form we can write

$$\xi(h) - I_{d_\xi} = \sum_{|\alpha|=1} D^{(\alpha)} \xi(e) q_\alpha(h) + O(|h|^2). \quad (3.3)$$

To estimate (3.2) we plug (3.3) into it. Then

$$\begin{aligned}& \text{Tr} \left((\xi(h) - I_{d_\xi})(\xi(h)^* - I_{d_\xi}^*) \widehat{f}(\xi)^* \widehat{f}(\xi) \right) \\ &= \text{Tr} \left(\sum_{|\alpha|=1, |\beta|=1} D^{(\alpha)} \xi(e) D^{(\beta)} \xi(e)^* \widehat{f}(\xi)^* \widehat{f}(\xi) q_\alpha(h) q_\beta(h) \right) + O(|h|^3).\end{aligned}$$

We observe that for each pair α, β the matrix $D^{(\alpha)} \xi(e) D^{(\beta)} \xi(e)^* + D^{(\beta)} \xi(e) D^{(\alpha)} \xi(e)^*$ entering this sum, is self-adjoint for every ξ , thus it can be written in the form

$$U^{-1} \Lambda_{\alpha, \beta, \xi} U = D^{(\alpha)} \xi(e) D^{(\beta)} \xi(e)^* + D^{(\beta)} \xi(e) D^{(\alpha)} \xi(e)^*,$$

where $\Lambda_{\alpha, \beta, \xi}$ is diagonal and U is unitary. Hence

$$\Lambda_{\alpha, \beta, \xi} = U(\sigma_{D^{(\alpha)}} \sigma_{D^{(\beta)}}^* + \sigma_{D^{(\beta)}} \sigma_{D^{(\alpha)}}^*) U^{-1},$$

and we can deduce that $\sum_{|\alpha|=1, |\beta|=1} \Lambda_{\alpha, \beta, \xi}$ is a diagonal positive second order elliptic symbol since $D^{(\alpha)}$'s can be chosen to give a basis of the Lie algebra of G , see [30, (10.25)]. The ellipticity is preserved by the action of an unitary matrix, due to the identities

$$\|B\|_{op} = \|BV\|_{op} = \|WB\|_{op},$$

where $B, V, W \in C^{d_\xi \times d_\xi}$ with V, W unitary.

Therefore, grouping terms with respect to the pair (α, β) and (β, α) , we have

$$\begin{aligned}
 & \operatorname{Tr} \left(\sum_{|\alpha|=1, |\beta|=1} D^{(\alpha)} \xi(e) D^{(\beta)} \xi(e)^* \widehat{f}(\xi)^* \widehat{f}(\xi) q_\alpha(h) q_\beta(h) \right) \\
 &= \operatorname{Tr} \left(\sum_{|\alpha|=1, |\beta|=1} \Lambda_{\alpha, \beta, \xi} [U \widehat{f}(\xi)^*] [U \widehat{f}(\xi)^*]^* q_\alpha(h) q_\beta(h) \right) \\
 &= \sum_{i=1}^{d_\xi} \left(\sum_{|\alpha|=1, |\beta|=1} (\Lambda_{\alpha, \beta, \xi})_{ii} ([U \widehat{f}(\xi)^*] [U \widehat{f}(\xi)^*]^*)_{ii} q_\alpha(h) q_\beta(h) \right) \\
 &\geq C|h|^2 \langle \xi \rangle^2 \sum_{i=1}^{d_\xi} (([U \widehat{f}(\xi)^*] [U \widehat{f}(\xi)^*]^*)_{ii}) \\
 &= C|h|^2 \langle \xi \rangle^2 \operatorname{Tr} ([U \widehat{f}(\xi)^*] [U \widehat{f}(\xi)^*]^*) \\
 &= C|h|^2 \langle \xi \rangle^2 \|\widehat{f}(\xi)\|_{\text{HS}}^2.
 \end{aligned}$$

We have proved that

$$\|\widehat{f}(\xi)(\xi(h) - I_{d_\xi})\|_{\text{HS}}^2 \geq C|h|^2 \langle \xi \rangle^2 \|\widehat{f}(\xi)\|_{\text{HS}}^2 \quad (3.4)$$

for small enough $|h|$. By (3.1) we have

$$\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq \frac{1}{|h|}} d_\xi^{q(\frac{2}{q}-\frac{1}{2})} \|\widehat{f}_h(\xi) - \widehat{f}(\xi)\|_{\text{HS}}^q = O(|h|^{\alpha q}),$$

and hence by (3.4) we obtain

$$\begin{aligned}
 & |h|^q \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq \frac{1}{|h|}} d_\xi^{q(\frac{2}{q}-\frac{1}{2})} \langle \xi \rangle^q \|\widehat{f}(\xi)\|_{\text{HS}}^q \\
 &\lesssim \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq \frac{1}{|h|}} d_\xi^{q(\frac{2}{q}-\frac{1}{2})} \|\widehat{f}_h(\xi) - \widehat{f}(\xi)\|_{\text{HS}}^q \leq O(|h|^{\alpha q}).
 \end{aligned}$$

Therefore, we obtain

$$\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq \frac{1}{|h|}} d_\xi^{q(\frac{2}{q}-\frac{1}{2})} \langle \xi \rangle^q \|\widehat{f}(\xi)\|_{\text{HS}}^q = O(|h|^{(\alpha-1)q}). \quad (3.5)$$

Now, for $\beta \leq q$ and $N \in \mathbb{N}$ we write

$$\Phi(N) = \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} d_{\xi}^{\beta(\frac{2}{\beta} - \frac{1}{2})} \langle \xi \rangle^{\beta} \|\widehat{f}(\xi)\|_{\text{HS}}^{\beta},$$

so that the analysis for $|h| \rightarrow 0$ will be translated to $N \rightarrow \infty$. In order to apply the Hölder inequality and for the forthcoming analysis it is convenient to write $\Phi(N)$ in the form

$$\Phi(N) = \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} d_{\xi}^2 \left(\frac{\langle \xi \rangle \|\widehat{f}(\xi)\|_{\text{HS}}}{\sqrt{d_{\xi}}} \right)^{\beta} = \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} a_{\xi} b_{\xi},$$

where $a_{\xi} = d_{\xi}^{2(\frac{\beta}{q})} \left(\frac{\langle \xi \rangle \|\widehat{f}(\xi)\|_{\text{HS}}}{\sqrt{d_{\xi}}} \right)^{\beta}$ and $b_{\xi} = d_{\xi}^{2(1-\frac{\beta}{q})}$.

Then, by Hölder inequality, (3.5) and (2.10) with $r = 0$, we get

$$\begin{aligned} \Phi(N) &\leq \left(\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} d_{\xi}^2 \left(\frac{\langle \xi \rangle \|\widehat{f}(\xi)\|_{\text{HS}}}{\sqrt{d_{\xi}}} \right)^q \right)^{\frac{\beta}{q}} \left(\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} d_{\xi}^2 \right)^{1-\frac{\beta}{q}} \\ &= O(N^{(1-\alpha)\beta}) N^{n(1-\frac{\beta}{q})}. \end{aligned}$$

Therefore

$$\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} d_{\xi}^2 \left(\frac{\langle \xi \rangle \|\widehat{f}(\xi)\|_{\text{HS}}}{\sqrt{d_{\xi}}} \right)^{\beta} = O(N^{(1-\alpha)\beta + n(1-\frac{\beta}{q})}). \quad (3.6)$$

Since $\widehat{(I - \mathcal{L}_G)^{\frac{1}{2}} f}(\xi) = \langle \xi \rangle \widehat{f}(\xi)$, we have proved that $\widehat{(I - \mathcal{L}_G)^{\frac{1}{2}} f} \in \ell^{\beta}(\widehat{G})$ provided that $(1 - \alpha)\beta + n(1 - \frac{\beta}{q}) \leq 0$, a condition which is equivalent to $\beta \geq \frac{n}{\alpha + n - \frac{n}{p} - 1}$ as we have assumed. This proves the first conclusion.

Now, an application of Lemma 3.1 to $H(\xi) = \widehat{f}(\xi)$ shows that $\widehat{f} \in \ell^{\gamma}(\widehat{G})$ for $\gamma > \frac{n\beta_0}{n+\beta_0}$ with $\beta_0 = \frac{n}{\alpha + n - \frac{n}{p} - 1}$. Since

$$\frac{n \left(\frac{n}{\alpha + n - \frac{n}{p} - 1} \right)}{\frac{n}{\alpha + n - \frac{n}{p} - 1} + n} = \frac{np}{\alpha p + np - n}$$

we conclude the proof. \square

Remark 3.3 In the case $G = \mathbb{T}^1$ we observe that Theorem 3.2 for the first conclusion takes the form:

$$f \in \text{Lip}_{\mathbb{T}^1}(\alpha; p) \implies \widehat{(I - \Delta)^{\frac{1}{2}} f} \in \ell^\beta(\mathbb{Z})$$

for $\frac{p}{p\alpha-1} < \beta$.

For the second conclusion:

$$f \in \text{Lip}_{\mathbb{T}^1}(\alpha; p) \implies \widehat{f} \in \ell^\gamma(\mathbb{Z})$$

for $\frac{p}{\alpha p + p - 1} < \gamma \leq q$.

Consequently Theorem 3.2 with the second conclusion is an extension of Titchmarsh theorem (cf. Titchmarsh [37, Theorem 84]) and Younis [39, Theorem 2.6]) to the setting of compact Lie groups. As it was already mentioned in the introduction the lower bound $\frac{p}{\alpha p + p - 1}$ in Theorem 3.2 is in general sharp as a counterexample can be constructed for $p = 2$ in the case $G = \mathbb{T}^1$.

For the proof of the second Titchmarsh theorem we will be using the following lemma due to Duren (cf. [9, p. 101]):

Lemma 3.4 Suppose $c_k \geq 0$ and $0 < b < a$. Then

$$\sum_{k=1}^N k^a c_k = O(N^b) \text{ as } N \rightarrow \infty$$

if and only if

$$\sum_{k=N}^{\infty} c_k = O(N^{b-a}) \text{ as } N \rightarrow \infty.$$

We can now state our second main result which extends Theorem B to compact Lie groups and consequently, in view of Lemma 2.4, to compact homogeneous manifolds (Theorem B').

Theorem 3.5 Let $0 < \alpha \leq 1$ and $f \in L^2(G)$. Then $f \in \text{Lip}_G(\alpha; 2)$ if and only if

$$\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \geq N} d_\xi \|\widehat{f}(\xi)\|_{\text{HS}}^2 = O(N^{-2\alpha}) \text{ as } N \rightarrow \infty. \quad (3.7)$$

Proof We first assume that $f \in \text{Lip}_G(\alpha; 2)$. By (3.5) applied to the case $p = q = 2$ we obtain

$$\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} d_\xi \langle \xi \rangle^2 \|\widehat{f}(\xi)\|_{\text{HS}}^2 = O(N^{2(1-\alpha)}). \quad (3.8)$$

Keeping the notations from the proof of Theorem 3.2 for $\Phi(N)$ with $\beta = 2$ we write

$$\Phi(N) = \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} d_\xi^2 \langle \xi \rangle^2 \left(\frac{\|\widehat{f}(\xi)\|_{\text{HS}}}{\sqrt{d_\xi}} \right)^2 = \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} \sum_{i,j=1}^{d_\xi} \langle \xi \rangle^2 \left(\frac{\|\widehat{f}(\xi)\|_{\text{HS}}}{\sqrt{d_\xi}} \right)^2.$$

We put $b_\xi = \left(\frac{\|\widehat{f}(\xi)\|_{\text{HS}}}{\sqrt{d_\xi}} \right)^2$ and in order to estimate the sum $\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} \sum_{i,j=1}^{d_\xi} \langle \xi \rangle^2 b_\xi$ we consider the set $W = \{\langle \xi \rangle : [\xi] \in \widehat{G}\}$ which is nothing but the range of eigenvalues of the operator $(I - \mathcal{L}_G)^{\frac{1}{2}}$ since by definition $\langle \xi \rangle := (1 + \lambda_{[\xi]}^2)^{\frac{1}{2}}$ where $\lambda_{[\xi]}^2$ are the eigenvalues of $-\mathcal{L}_G$. Since W is discrete countable we can enumerate it as $W = \{\lambda_k : k \in \mathbb{N}\}$. Of course this enumeration also induces an enumeration of \widehat{G} and by the eigenvalue estimate $\lambda_k \approx k^{\frac{1}{n}}$ (see e.g [1]) we obtain

$$\begin{aligned} \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} \sum_{i,j=1}^{d_\xi} \langle \xi \rangle^2 b_\xi &= \sum_{\lambda_k \leq N} \sum_{i,j=1}^{d_k} \lambda_k^2 b_k \\ &\approx \sum_{k \leq N^n} \sum_{i,j=1}^{d_k} k^{\frac{2}{n}} b_k \\ &= \sum_{k \leq N^n} k^{\frac{2}{n}} c_k, \end{aligned} \quad (3.9)$$

with $c_k = d_k^2 b_k$. We now apply (3.8) to get

$$\sum_{k \leq N^n} k^{\frac{2}{n}} c_k = \sum_{k=1}^{N_0} k^{\frac{2}{n}} c_k = O(N_0^{\frac{2(1-\alpha)}{n}}),$$

where $N_0 = N^n$. An application of Duren's Lemma 3.4 gives us

$$\sum_{k=N_0}^{\infty} c_k = O\left(N_0^{\frac{2(1-\alpha)}{n} - \frac{2}{n}}\right) = O\left(N_0^{-\frac{2\alpha}{n}}\right).$$

Hence

$$\sum_{k \geq N^n} c_k = \sum_{\lambda_k \geq N} d_k^2 b_k = O(N^{-2\alpha}).$$

Therefore

$$\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \geq N} d_\xi \|\widehat{f}(\xi)\|_{\text{HS}}^2 = \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \geq N} d_\xi^2 \left(\frac{\|\widehat{f}(\xi)\|_{\text{HS}}}{\sqrt{d_\xi}} \right)^2 = O(N^{-2\alpha}),$$

which shows (3.7).

We now assume (3.7). By Plancherel theorem and with the notation for f_h as left translation as in the proof of Theorem 3.2 we have

$$\|\widehat{f}(\xi)(\xi(h) - I_{d_\xi})\|_{\ell^2(\widehat{G})}^2 = \|f_h - f\|_{L^2(G)}^2. \quad (3.10)$$

Following the analysis of the proof of Theorem 3.2 we note that from the ellipticity argument used to obtain (3.4), we can now just use the fact that $\sum_{|\alpha|=1, |\beta|=1} \Lambda_{\alpha, \beta, \xi}$ is of the second order. Then we get

$$\|\widehat{f}(\xi)(\xi(h) - I_{d_\xi})\|_{\text{HS}}^2 \leq C|h|^2 \langle \xi \rangle^2 \|\widehat{f}(\xi)\|_{\text{HS}}^2. \quad (3.11)$$

Therefore the proof is now reduced to estimate

$$\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \geq N} d_\xi \langle \xi \rangle^2 \|\widehat{f}(\xi)\|_{\text{HS}}^2.$$

But an analogous argument as in the above proof of the “only if” part and using the same notations yield

$$\begin{aligned} \|f_h - f\|_{L^2(G)}^2 &\leq |h|^2 \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \geq N} d_\xi \langle \xi \rangle^2 \|\widehat{f}(\xi)\|_{\text{HS}}^2 \\ &\approx |h|^2 \sum_{k=N}^{\infty} c_k = |h|^2 O(N^{2(1-\alpha)}) = O(|h|^{2\alpha}), \end{aligned}$$

with $N = \frac{1}{|h|}$, which concludes the proof. \square

We now formulate an application of Theorem 3.5 to the regularity of Fourier multipliers on Hölder spaces mentioned in Corollary 1.2. For convenience of the reader we repeat its statement:

Corollary 3.6 *Let $0 \leq \gamma < 1$ and let $a : \widehat{G} \rightarrow \bigcup_{d \in \mathbb{N}} \mathbb{C}^{d \times d}$ be such that $a(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ for each ξ and*

$$\|a(\xi)\|_{op} \leq C \langle \xi \rangle^{-\gamma}.$$

Let A be the Fourier multiplier with symbol a , i.e. $\widehat{Af}(\xi) = a(\xi)\widehat{f}(\xi)$ for all $\xi \in \widehat{G}$. Then

$$A : \text{Lip}_G(\alpha; 2) \rightarrow \text{Lip}_G(\alpha + \gamma; 2)$$

is bounded, for all α such that $0 < \alpha < 1 - \gamma$.

Proof Let $f \in \text{Lip}_G(\alpha; 2)$. Then by Theorem 3.5 we have

$$\begin{aligned}
 \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \geq N} d_\xi \|\widehat{Af}(\xi)\|_{\text{HS}}^2 &= \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \geq N} d_\xi \|a(\xi)\widehat{f}(\xi)\|_{\text{HS}}^2 \\
 &\leq \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \geq N} d_\xi \|a(\xi)\|_{op}^2 \|\widehat{f}(\xi)\|_{\text{HS}}^2 \\
 &\leq C \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \geq N} d_\xi \langle \xi \rangle^{-2\gamma} \|\widehat{f}(\xi)\|_{\text{HS}}^2 \\
 &\leq CN^{-2\gamma} \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \geq N} d_\xi \|\widehat{f}(\xi)\|_{\text{HS}}^2 \\
 &= O(N^{-2(\alpha+\gamma)}) \text{ as } N \rightarrow \infty.
 \end{aligned}$$

Again by Theorem 3.5 this implies that $A : \text{Lip}_G(\alpha; 2) \rightarrow \text{Lip}_G(\alpha + \gamma; 2)$ is bounded for all $\alpha > 0$ such that $\alpha + \gamma < 1$. \square

As an example we will consider the Lipschitz-Sobolev regularity for Bessel potential operators on compact Lie groups. First we observe that if $A = (I - \mathcal{L}_G)^{-\frac{\gamma}{2}}$ with $0 \leq \gamma < 1$, by Corollary 3.6 we have

$$\|(I - \mathcal{L}_G)^{-\frac{\gamma}{2}} f\|_{\text{Lip}_G(\alpha+\gamma;2)} \leq C \|f\|_{\text{Lip}_G(\alpha;2)}$$

for all α such that $0 < \alpha < 1 - \gamma$.

Hence

$$\|f\|_{\text{Lip}_G(\alpha+\gamma;2)} \leq C \|(I - \mathcal{L}_G)^{\frac{\gamma}{2}} f\|_{\text{Lip}_G(\alpha;2)}. \quad (3.12)$$

We can now introduce the Sobolev–Lipschitz space $H^\gamma \text{Lip}_G(\alpha, 2)$ for every $0 \leq \gamma < 1$ and $0 < \alpha < 1 - \gamma$ by

$$H^\gamma \text{Lip}_G(\alpha, 2) := \left\{ f \in \mathcal{D}'(G) : (I - \mathcal{L}_G)^{\frac{\gamma}{2}} f \in \text{Lip}_G(\alpha, 2) \right\}.$$

From (3.12) we obtain:

Corollary 3.7 *For every $0 \leq \gamma < 1$ and $0 < \alpha < 1 - \gamma$ we have the continuous embedding*

$$H^\gamma \text{Lip}_G(\alpha, 2) \hookrightarrow \text{Lip}_G(\alpha + \gamma, 2).$$

4 Dini–Lipschitz functions

So far we have been dealing with the Hölder–Lipschitz condition on compact homogeneous manifolds in the L^p setting and a characterisation in the L^2 case. In this

section we will consider a different condition, the so-called Dini–Lipschitz condition on L^2 and we will generalise the corresponding Titchmarsh theorems (cf. [37, Theorem 85]).

We shall first establish an extension of Duren’s lemma (cf. [9, p. 101]), Lemma 3.4 in this paper, adapted to the Dini–Lipschitz condition.

Lemma 4.1 *Suppose $d \in \mathbb{R}$, $c_k \geq 0$ and $0 < b < a$. Then*

$$\sum_{k=1}^N k^a c_k = O(N^b (\log N)^d) \quad \text{as } N \rightarrow \infty$$

if and only if

$$\sum_{k=N}^{\infty} c_k = O(N^{b-a} (\log N)^d) \quad \text{as } N \rightarrow \infty.$$

Proof We first assume that $\sum_{k=1}^N k^a c_k = O(N^b (\log N)^d)$ as $N \rightarrow \infty$, and write

$$S_N = \sum_{k=1}^N k^a c_k. \quad (4.1)$$

By using the partial summation formula (cf. [28, Theorem 3.41]) and (4.1) we have

$$\begin{aligned} \sum_{k=N}^M c_k &= \sum_{k=N}^{M-1} S_k (k^{-a} - (k+1)^{-a}) + S_M M^{-a} - S_{N-1} N^{-a} \\ &\leq C \sum_{k=N}^{M-1} k^b (\log k)^d (k^{-a} - (k+1)^{-a}) + C M^{b-a} (\log M)^d \\ &\leq C \sum_{k=N}^{M-1} k^{b-a-1} (\log k)^d + C M^{b-a} (\log M)^d, \end{aligned}$$

where we have also used $k^{-a} - (k+1)^{-a} \leq ck^{-a-1}$ in the last inequality.

Letting $M \rightarrow \infty$ the conclusion is reduced to estimate the series $\sum_{k=N}^{\infty} k^{b-a-1} (\log k)^d$. We observe that an integration by parts argument shows that

$$\begin{aligned} \int_N^{\infty} t^{b-a-1} (\log t)^d dt &= \frac{1}{b-a} \int_N^{\infty} (\log t)^d \frac{d}{dt} t^{b-a} dt \\ &= \frac{t^{b-a}}{b-a} (\log t)^d \Big|_N^{\infty} - \frac{d}{b-a} \int_N^{\infty} t^{b-a-1} (\log t)^{d-1} dt \\ &= \frac{N^{b-a}}{a-b} (\log N)^d + \frac{d}{a-b} \int_N^{\infty} t^{b-a-1} (\log t)^{d-1} dt. \end{aligned} \quad (4.2)$$

We now distinguish two cases: $d \leq 0$ and $d > 0$. First, if $d \leq 0$ we obtain

$$\begin{aligned} \int_N^\infty t^{b-a-1} (\log t)^d dt &= \frac{N^{b-a}}{a-b} (\log N)^d - C_{N,a,b,d} \\ &= O(N^{b-a} (\log N)^d) \text{ as } N \rightarrow \infty, \end{aligned}$$

where the number $C_{N,a,b,d}$ is non-negative since $\frac{d}{a-b} \leq 0$ and $t^{b-a-1} (\log t)^{-d-1} > 0$.

If $d > 0$, we take the least positive integer ℓ such that $d \leq \ell - 1$. An inductive argument applying integration by parts to the integral on the right-hand side of (4.2) gives us

$$\begin{aligned} \int_N^\infty t^{b-a-1} (\log t)^d dt &= O(N^{b-a} (\log N)^d) \\ &\quad + \frac{d(d-1) \cdots (d-(\ell-1))}{(a-b)^\ell} \int_N^\infty t^{b-a-1} (\log t)^{d-\ell} dt. \end{aligned}$$

Once more, the fact that $t^{b-a-1} (\log t)^{d-\ell} \geq 0$ and the choice of ℓ implies that

$$d(d-1) \cdots (d-(\ell-1)) \leq 0,$$

which concludes the proof of the “only if part”.

Conversely, we assume the second assertion and write

$$R_N = \sum_{k=N}^\infty c_k. \quad (4.3)$$

Now, the partial summation formula gives

$$\begin{aligned} \sum_{k=1}^N k^a c_k &= \sum_{k=1}^{N-1} R_k (k^a - (k+1)^a) + R_N N^a \\ &\leq C \sum_{k=1}^{N-1} k^{b-a} (\log k)^d (k^a - (k+1)^a) + C N^{b-a} (\log(N))^d N^a \\ &\leq C \sum_{k=1}^{N-1} k^{b-a} (\log k)^d k^{a-1} + C N^b (\log(N))^d \\ &= C \sum_{k=1}^{N-1} k^{b-1} (\log k)^d + C N^b (\log(N))^d. \end{aligned}$$

The sum $\sum_{k=1}^{N-1} k^{b-1}(\log k)^d$ can be estimated by integration in a similar way as we have obtained (4.2). The inductive argument explained above gives

$$\int_1^{N-1} t^{b-1}(\log t)^d dt = O((N-1)^b(\log(N-1))^d) \\ + \frac{d(d-1)\cdots(d-(\ell-1))}{b^\ell} \int_1^{N-1} t^{b-1}(\log t)^{d-\ell} dt.$$

By using this formula, one can deduce that

$$\sum_{k=1}^N k^a c_k = O(N^b(\log N)^d) \text{ as } N \rightarrow \infty.$$

The proof is complete. \square

We can now establish our first theorem on Dini–Lipschitz functions.

Theorem 4.2 *Let G/K be a compact homogeneous manifold. Let $\alpha \geq 0$ and $d \in \mathbb{R}$. Then the conditions*

$$\|f(h\cdot) - f(\cdot)\|_{L^2(G/K)} = O\left(|h|^\alpha \left(\log \frac{1}{|h|}\right)^d\right) \text{ as } |h| \rightarrow 0 \quad (4.4)$$

and

$$\sum_{[\xi] \in \widehat{G}_0, \langle \xi \rangle \geq N} d_\xi \|\widehat{f}(\xi)\|_{\text{HS}}^2 = O\left(N^{-2\alpha}(\log N)^{2d}\right) \text{ as } N \rightarrow \infty \quad (4.5)$$

are equivalent.

Proof As in the previous section it will be enough to prove our results for compact Lie groups. First we recall that by (3.10) we have

$$\|\widehat{f}(\xi)(\xi(h) - I_{d_\xi})\|_{\ell^2(\widehat{G})}^2 = \|f_h - f\|_{L^2(G)}^2. \quad (4.6)$$

Now if we assume (4.4) we obtain

$$\|\widehat{f}(\xi)(\xi(h) - I_{d_\xi})\|_{\ell^2(\widehat{G})}^2 = O\left(|h|^{2\alpha} \left(\log \frac{1}{|h|}\right)^{2d}\right) \text{ as } |h| \rightarrow 0. \quad (4.7)$$

A look at the deduction of (3.5) and by (4.7) shows that for $p = q = 2$:

$$|h|^2 \sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq \frac{1}{|h|}} d_\xi \langle \xi \rangle^2 \|\widehat{f}(\xi)\|_{\text{HS}}^2 = O(|h|^{2\alpha}(\log(1/|h|))^{2d}) \text{ as } |h| \rightarrow 0. \quad (4.8)$$

Hence

$$\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq \frac{1}{|h|}} d_{\xi} \langle \xi \rangle^2 \|\widehat{f}(\xi)\|_{\text{HS}}^2 = O(|h|^{2\alpha-2} (\log |h|)^{2d}) \text{ as } |h| \rightarrow 0.$$

We now write $N = 1/h$. Then

$$\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} d_{\xi} \langle \xi \rangle^2 \|\widehat{f}(\xi)\|_{\text{HS}}^2 = O(N^{2-2\alpha} (\log(N))^{2d}) \text{ as } N \rightarrow \infty. \quad (4.9)$$

An analogous argument as in the proof of Theorem 3.5 to obtain (3.9) and using the same notations lead us to

$$\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} d_{\xi} \langle \xi \rangle^2 \|\widehat{f}(\xi)\|_{\text{HS}}^2 \leq \sum_{k \leq N^n} k^{\frac{2}{n}} c_k = \sum_{k=1}^{N_0} k^{\frac{2}{n}} c_k = O(N_0^{\frac{2-2\alpha}{n}} (\log N_0^{\frac{1}{n}})^{2d}),$$

where $N_0 = N^{\frac{1}{n}}$. An application of Lemma 4.1 gives us

$$\sum_{k=N_0}^{\infty} c_k = O\left(N_0^{\frac{2-2\alpha}{n} - \frac{2}{n}} (\log N_0^{\frac{1}{n}})^{2d}\right) \text{ as } N_0 \rightarrow \infty.$$

Therefore

$$\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \geq N} d_{\xi} \|\widehat{f}(\xi)\|_{\text{HS}}^2 = O(N^{-2\alpha} (\log N)^{2d}) \text{ as } N \rightarrow \infty.$$

A look at the proof above shows that the converse also holds. \square

To end this work we now give a Dini–Lipschitz version of Theorem 3.2.

Theorem 4.3 *Let G/K be a compact homogeneous manifold of dimension n . Let $0 < \alpha \leq 1$, $d \in \mathbb{R}$, $1 < p \leq 2$, and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that*

$$\|f(h \cdot) - f(\cdot)\|_{L^p(G/K)} = O\left(|h|^{\alpha} \left(\log \frac{1}{|h|}\right)^d\right) \text{ as } |h| \rightarrow 0. \quad (4.10)$$

Then we have

$$\widehat{(I - \mathcal{L}_G)^{\frac{1}{2}} f} \in \ell^{\beta}(\widehat{G}_0),$$

provided that either

$$\frac{n}{\alpha + n - \frac{n}{p} - 1} < \beta \leq q \quad \text{for any } d \in \mathbb{R}$$

or

$$\frac{n}{\alpha + n - \frac{n}{p} - 1} \leq \beta \leq q \quad \text{for } d \leq 0.$$

Consequently, in both cases we have

$$\widehat{f} \in \ell^\gamma(\widehat{G}_0) \quad \text{for } \frac{np}{\alpha p + np - n} < \gamma \leq q.$$

Proof Arguing as in the proof of Theorem 3.2, under assumptions of Theorem 4.3 instead of (3.6) we arrive at

$$\sum_{[\xi] \in \widehat{G}, \langle \xi \rangle \leq N} d_\xi^2 \left(\frac{\langle \xi \rangle \|\widehat{f}(\xi)\|_{\text{HS}}}{\sqrt{d_\xi}} \right)^\beta = O(N^{(1-\alpha)\beta + n(1-\frac{\beta}{q})} (\log N)^{d\beta}) \text{ as } N \rightarrow \infty. \quad (4.11)$$

So we get

$$\widehat{(I - \mathcal{L}_G)^{\frac{1}{2}} f} \in \ell^\beta(\widehat{G}_0)$$

provided that $(1 - \alpha)\beta + n(1 - \frac{\beta}{q}) < 0$, or $(1 - \alpha)\beta + n(1 - \frac{\beta}{q}) \leq 0$ and $d \leq 0$. The last conclusion follows in the same way as in Theorem 3.2. \square

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